Available online at www.sciencedirect.com
ScienceDirect
Journal of Geometry and Physics 57 (2007) 913–923
JOURNAL OF GEOMETRY AND PHYSICS
www.elsevier.com/locate/jgp

Spacelike surfaces with positive definite second fundamental form in 3D spacetimes

J.A. Aledo[a], S. Haesen[b,1], A. Romero[c,*]

[a] E.P.S. de Albacete, Universidad de Castilla-La Mancha, 02071 Albacete, Spain
[b] Department of Mathematics, Katholieke Universiteit Leuven, Celestijnenlaan 200B, 3001 Heverlee, Belgium
[c] Dep. de Geometría y Topología, Universidad de Granada, 18071 Granada, Spain

Received 16 June 2005; received in revised form 29 April 2006; accepted 4 July 2006
Available online 8 August 2006

Abstract

For a spacelike surface with positive definite second fundamental form in any 3-dimensional Lorentzian manifold, a new formula relating its mean and Gauss curvature with the Gauss curvature of the second fundamental form is obtained. As an application, necessary and sufficient conditions are established in order to prove that such a compact spacelike surface is totally umbilical.

MSC: 53C42; 53C50; 53C80

Keywords: Spacetime; Compact spacelike surface; Positive definite second fundamental form; Gauss curvature

1. Introduction

It is commonly argued that (connected) 3-dimensional time-oriented Lorentzian manifolds (3D spacetimes, in brief) are too unrealistic to give much insight into usual 4-dimensional relativistic models. However, they have been deeply studied from a purely geometric viewpoint and clearly present great mathematical interest. Moreover, 3D spacetimes are useful to explore the foundations of classical and quantum gravity (see [7] and references therein). It is surprising that relativity on 3D spacetimes is simpler than in four dimensions and becomes a topological field theory [6]. Sometimes, 3D spacetimes even provide us with valuable information for relativistic models; in fact, relevant results of the problem of the reduction of the 4-dimensional Einstein equations, formulated on circle bundles over certain 3D spacetimes, have been found [13].

Spacelike hypersurfaces play an important role in spacetimes. Among these hypersurfaces, the totally umbilical ones, i.e. those whose shape operator is a multiple of the identity transformation, have a rich geometry. In fact, the study of this family of hypersurfaces lies into the conformal geometry of the spacetime, because a pointwise conformal

* Corresponding author. Tel.: +34 958 24 33 66; fax: +34 958 24 32 81.
E-mail addresses: juanangel.aledo@uclm.es (J.A. Aledo), stefan.haesen@wis.kuleuven.be (S. Haesen), aromero@ugr.es (A. Romero).
[1] Postdoctoral researcher of the F.W.O.-Vlaanderen.

0393-0440/$ - see front matter
doi:10.1016/j.geomphys.2006.07.002

change of the ambient metric preserves the character of being totally umbilical of a spacelike hypersurface. The second fundamental form (with respect to the unit normal vector field in the time-orientation of the spacetime) of a (non-totally geodesic) totally umbilical hypersurface is obviously positive definite and thus, it provides the hypersurface with a new Riemannian metric. It is an obvious remark that, in general, a conformal change of the spacetime metric does not preserve the character of being positive definite of the second fundamental form of a spacelike surface. But, it is not difficult to see that if we have a spacelike surface M with positive definite second fundamental form in a 3D spacetime $\bar{M}$, then a pointwise conformal change of the metric of $\bar{M}$ with a smooth function such that its gradient is timelike in the same time-orientation of $\bar{M}$, provides M with a positive definite second fundamental form with respect to the new conformal metric. Geometrically, if a spacelike hypersurface has positive definite second fundamental form, then the future-directed timelike geodesics orthogonal to the hypersurface M are really spreading out near M. Thus, in particular, the volume of M does increase when it is compact. Therefore, the existence of a compact spacelike hypersurface with positive definite second fundamental form means that the spacetime is really expanding.

The main objective of this paper is to obtain several characterizations for totally umbilical spacelike surfaces, among the class of spacelike surfaces with positive definite second fundamental form, in a 3D spacetime. Pursuing this aim, we develop a new formula for spacelike surfaces with positive definite second fundamental form in a general 3D spacetime $\bar{M}$, Theorem 3.4. This formula relates the mean and Gauss curvatures of the induced metric or first fundamental form, the Gauss curvature of the second fundamental form and the difference tensor of the two associated Levi-Civita connections.

Similar formulas have been used several times in the past (see for instance [2,3,9,11,14,15]) but always when an ambient space of constant sectional curvature was considered. This assumption simplifies enormously the computations because the shape operator is then a Codazzi tensor in the classical sense. However, our spacetimes have non-constant sectional curvature in general and consequently the Codazzi equation has a non-zero curvature term in general. This means there is a great difference in our starting point with respect to previous papers in that direction. In fact, this technical difficulty now produces extra terms in our formula which are not easy to handle and also complicate its later use.

Although the formula is developed considering the only assumption that the spacelike surface has positive definite second fundamental form (Section 3), we use it as a tool to study compact spacelike surfaces. It is worth pointing out that in a general 3D spacetime there may not exist either surfaces with positive definite second fundamental form (see Proposition 2.1 and the comment thereafter), or compact ones; for instance, when $\bar{M}$ is a generalized Robertson–Walker spacetime, it must be necessarily spatially closed (see [4] for details). Even more, not in every spatially closed generalized Robertson–Walker spacetime does there exist compact spacelike surfaces with positive definite second fundamental form. In fact, a simple reasoning using the Gauss–Bonnet theorem allows us to see that this occurs when the fiber is a 2-dimensional torus with its canonical metric, $I = \mathbf{R}$, and the warping function is $f = 1$. Section 4 is devoted to provide several characterizations of totally umbilical spacelike surfaces. We find additional hypotheses for a compact surface with positive definite second fundamental form to be a totally umbilical spacelike surface. Thus, from Theorem 4.1 to Theorem 4.6 we consider assumptions related to the difference tensor L between the Levi-Civita connections for the first and second fundamental forms of the surface.

2. Preliminaries

Let $(\bar{M}, \bar{g})$ be a 3D spacetime. A smooth immersion $x : M \to \bar{M}$ of a (connected) 2-dimensional manifold M is said to be a spacelike surface if the induced metric on M via x, $x^\star(\bar{g})$, is Riemannian, which as usual we will denote by $\bar{g}$. If M is an immersed spacelike surface in $\bar{M}$, the time-orientability of $\bar{M}$ allows us to define $N \in \mathfrak{X}^\perp(M)$ as the only unit timelike vector field normal to M in the same time-orientation as $\bar{M}$. If A stands for the shape operator of M in $\bar{M}$ associated to N, we will denote the mean curvature of M by $H = -(1/2)\mathrm{tr}\,(A)$. The choice of the minus sign in the definition of H is motivated by the fact that in this case the mean curvature vector field is given by $\vec{H} = HN$, and therefore, $H(p) > 0$, $p \in M$, if and only if $\vec{H}(p)$ is in the time-orientation determined by $N(p)$.

The Riemann curvature tensors R and $\bar{R}$ of M and $\bar{M}$ respectively are related by the Gauss equation

$$\bar{g}(R(X,Y)Z,W) = \bar{g}(\bar{R}(X,Y)Z,W) - \bar{g}(AX,Z)\bar{g}(AY,W) + \bar{g}(AY,Z)\bar{g}(AX,W), \tag{1}$$

for any $X, Y, Z, W \in \mathfrak{X}(M)$. The Codazzi equation of the surface reads

$$(\nabla_X A)Y - (\nabla_Y A)X = \bar{R}(X,Y)N, \tag{2}$$

for any $X, Y \in \mathfrak{X}(M)$, where ∇ is the Levi-Civita connection of the induced metric on M. Observe that the right-hand side of (2) vanishes if the ambient space has constant sectional curvature. In the literature, a Codazzi tensor is a symmetric (1,1)-tensor field B which satisfies $(\nabla_X B)Y = (\nabla_Y B)X$. Thus, in general, we are not dealing with Codazzi tensors in this sense.

From (1) it follows that

$$2K = \bar{S} + 2\,\overline{\mathrm{Ric}}(N, N) + \mathrm{tr}\,(A^2) - (\mathrm{tr}\,A)^2,$$

where K is the Gauss curvature of M and $\bar{S}$ the scalar curvature of $\bar{M}$. We further have that the sectional curvature $\bar{K}(p)$ of each tangent plane $\mathrm{d}x_p(T_pM)$ in $\bar{M}$, satisfies $\bar{K} = \frac{1}{2}\bar{S} + \overline{\mathrm{Ric}}(N, N)$. Then, using the characteristic equation for the shape operator, $A^2 - \mathrm{tr}\,(A)A + \det(A)I = 0$, we obtain the following expression for the Gauss–Kronecker curvature of the surface M,

$$\det(A) = \bar{K} - K. \tag{3}$$

When $\det(A) > 0$, the second fundamental form II is given by

$$\mathrm{II}(X, Y) = -\bar{g}(AX, Y), \tag{4}$$

where $X, Y \in \mathfrak{X}(M)$, determines a definite metric on M. In fact, we can suppose (up to a change of orientation) that II is positive definite, i.e., II is a Riemannian metric on M. From (3) we thus obtain a necessary and sufficient condition for II to be positive definite.

Proposition 2.1. *On a spacelike surface M in a* 3D *spacetime $\bar{M}$ the following two conditions are equivalent:*

(i) II *is a positive definite metric on M (up to a choice of orientation),*
(ii) *The Gauss curvature K of M satisfies $K < \bar{K}$.*

Remark 2.2. This result gives an obstruction to the existence of a spacelike surface with positive definite second fundamental form in terms of the curvature of the ambient space. In fact, if $\bar{M}$ admits such a compact spacelike surface with the topology of $\mathbf{S}^2$, it follows from the Gauss–Bonnet theorem that the sectional curvature of $\bar{M}$ must be positive on some spacelike plane. □

Remark 2.3. Let M be a 2-dimensional compact manifold, $\bar{M}$ a 3D spacetime and consider E the set of spacelike immersions $x : M \to \bar{M}$ with positive definite second fundamental form. Suppose that E is non-empty. Since the condition $\det(A) > 0$ implies that the usual area functional $x \mapsto \mathrm{area}(M, \bar{g}_x)$ has no critical point in E, it would be of interest to know if the area functional with respect to the second fundamental form,

$$\mathcal{F}_{\mathrm{II}}(x) := \mathrm{area}\,(M, \mathrm{II}_x) = \int_M \mathrm{d}\Omega_{\mathrm{II}_x} = \int_M \sqrt{\det(A_x)}\,\mathrm{d}\Omega_x,$$

where II_x is the second fundamental form corresponding to x, has a critical point in E. It turns out that this is in general not the case. Let, for example, $x : M \to \mathbf{S}^3_1 \subset \mathbf{L}^4$ be a compact spacelike surface in the 3-dimensional de Sitter space of sectional curvature 1. Consider for $t \in \mathbf{R}$ the parallel surface $x_t : M \to \mathbf{S}^3_1 \subset \mathbf{L}^4$, which is given by

$$x_t(p) = \overline{\exp}_{x(p)}(tN(p)) = \cosh(t)x(p) + \sinh(t)N(p),$$

where $p \in M$, $\overline{\exp}$ denotes the exponential map in $\mathbf{S}^3_1$ and N the unit normal timelike vector field to M. From [1] we have that

$$(\mathrm{d}x_t)_p(v) = \mathrm{d}x_p(\cosh(t)v - \sinh(t)A_p(v)),$$

for any $p \in M$ and $v \in T_pM$, which implies that

$$\mathrm{d}\Omega_t = \{\cosh^2(t) - \mathrm{tr}\,(A)\sinh(t)\cosh(t) + \det(A)\sinh^2(t)\}\mathrm{d}\Omega. \tag{5}$$

We can then compute the variation of $\mathrm{d}\Omega_{\mathrm{II}_t} = \sqrt{\det(A_t)}\mathrm{d}\Omega_t$. A well-known result states that $\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\log(\det(A_t)) = \mathrm{tr}\,(A_t^{-1}A_t')|_{t=0}$. But, using [1] it follows that $(A_t')|_{t=0} = -I + A^2$, and thus

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \log(\det(A_t)) = -\mathrm{tr}\,(A^{-1}) + \mathrm{tr}\,(A).$$

From the characteristic equation for A it follows that $\mathrm{tr}\,(A^{-1}) = \frac{\mathrm{tr}\,(A)}{\det(A)}$. Combining these results we have

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \det(A_t) = \mathrm{tr}\,(A)[\det(A) - 1].$$

From (5) it follows further that $\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{d}\Omega_t)|_{t=0} = -\mathrm{tr}\,(A)\mathrm{d}\Omega$, and therefore we get

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathcal{F}_{\mathrm{II}}(x_t) = \int_M \frac{H\,[1+\det(A)]}{\det(A)}\mathrm{d}\Omega_{\mathrm{II}}.$$

Finally, since $\det(A)$ and H are strictly positive, it follows that the functional $\mathcal{F}_{\mathrm{II}}$ has no critical point in E if the ambient space is de Sitter. □

Now, let ∇^{II} denote the Levi-Civita connection with respect to II. The difference tensor L between the Levi-Civita connections ∇^{II} and ∇ is given by

$$L(X,Y) = {\nabla^{\mathrm{II}}}_X Y - \nabla_X Y, \tag{6}$$

for all $X, Y \in \mathfrak{X}(M)$. From the Koszul formula for ∇^{II}, and using the Codazzi equation (2), we find that

$$L(X,Y) = \frac{1}{2}A^{-1}\left\{(\nabla_X A)Y - \left[\bar{R}(X,N)Y\right]^T\right\}, \tag{7}$$

where $[\ ]^T$ denotes the tangent component to the surface M. Besides the obvious symmetry $L(X,Y) = L(Y,X)$, there also holds the relation

$$\mathrm{II}(L(X,Y),Z) - \mathrm{II}(L(X,Z),Y) = \bar{g}(\bar{R}(X,N)Y,Z). \tag{8}$$

We further need the following result.

Lemma 2.4. *Let M be a totally umbilical spacelike surface in a* 3D *spacetime $\bar{M}$, with* $\det(A) > 0$. *Then the trace of L with respect to* II *vanishes, i.e.,* $\mathrm{tr}_{\mathrm{II}}(L) = 0$.

Proof. Since M is totally umbilical there exists a smooth function, $\rho : M \to \mathbf{R}$ such that $A = \rho I$. Then $\mathrm{II}(X,Y) = -\rho\bar{g}(X,Y)$ for all $X, Y \in \mathfrak{X}(M)$ and therefore

$${\nabla^{\mathrm{II}}}_X Y - \nabla_X Y = \frac{1}{2\rho}\{X(\rho)Y + Y(\rho)X - \bar{g}(X,Y)\nabla\rho\}.$$

Thus

$$L(X,Y) = \frac{1}{2\rho}\left\{\mathrm{II}(X,\nabla^{\mathrm{II}}\rho)Y + \mathrm{II}(Y,\nabla^{\mathrm{II}}\rho)X - \mathrm{II}(X,Y)\nabla^{\mathrm{II}}\rho\right\},$$

where we have used the fact that $\nabla^{\mathrm{II}}\rho = -\frac{1}{\rho}\nabla\rho$. Then, if we denote by $\{V_1, V_2\}$ a II-orthonormal local frame, we have

$$\mathrm{tr}_{\mathrm{II}}(L) = \frac{1}{2\rho}\left\{2\sum_{i=1}^{2}\mathrm{II}(V_i,\nabla^{\mathrm{II}}\rho)V_i - 2\nabla^{\mathrm{II}}\rho\right\} = 0. \quad \square$$

3. A formula for the Gauss curvature of the metric II

On a spacelike surface M such that II is a Riemannian metric, the Riemann–Christoffel curvature tensor R^{II} of II satisfies

$$R^{\mathrm{II}}(X,Y)Z = R(X,Y)Z + Q_1(X,Y)Z + Q_2(X,Y)Z, \tag{9}$$

with

$$Q_1(X,Y)Z = ({\nabla^{\mathrm{II}}}_Y L)(X,Z) - ({\nabla^{\mathrm{II}}}_X L)(Y,Z),$$
$$Q_2(X,Y)Z = L(X,L(Y,Z)) - L(Y,L(X,Z)),$$

and $X, Y, Z \in \mathfrak{X}(M)$. Contracting in (9) we get

$$\mathrm{Ric}^{\mathrm{II}}(X,Y) = \mathrm{Ric}(X,Y) + \widehat{Q}_1(X,Y) + \widehat{Q}_2(X,Y), \tag{10}$$

where $\widehat{Q}_i(X,Y) = \mathrm{tr}\{Z \mapsto Q_i(X,Z)Y\}$, $i = 1, 2$. If we further take the trace in (10) with respect to the metric II, we find

$$2K^{\mathrm{II}} = \mathrm{tr}_{\mathrm{II}}(\mathrm{Ric}) + \mathrm{tr}_{\mathrm{II}}(\widehat{Q}_1) + \mathrm{tr}_{\mathrm{II}}(\widehat{Q}_2), \tag{11}$$

with K^{II} the Gauss curvature of the surface M with respect to the metric II. Here the trace of a $(0, 2)$-tensor T with respect to II is, as usual, the ordinary trace of the $(1, 1)$-tensor $\tilde{T}$ which is II-equivalent to T, i.e., defined by $\mathrm{II}(\tilde{T}(X), Y) = T(X, Y)$, for any $X, Y \in \mathfrak{X}(M)$.

Lemma 3.1. *The trace with respect to* II *of the Ricci tensor* Ric *is given by*

$$\mathrm{tr}_{\mathrm{II}}(\mathrm{Ric}) = -\frac{\mathrm{tr}\,(A)}{\det(A)}K.$$

Proof. From the Gauss equation (1) for the spacelike surface M of $\bar{M}$, we have

$$\mathrm{Ric}(X,Y) = \overline{\mathrm{Ric}}(X,Y) + \bar{g}(\bar{R}(X,N)Y,N) - \mathrm{tr}\,(A)\bar{g}(AX,Y) + \bar{g}(AX,AY),$$

for any $X, Y \in \mathfrak{X}(M)$. Now, choose a $\bar{g}$-orthonormal local frame $\{e_1, e_2\}$ which diagonalizes A, i.e., $Ae_i = \lambda_i e_i$, $i = 1, 2$. Then, $\{V_1, V_2\}$, with

$$V_i = \frac{1}{\sqrt{-\lambda_i}}\, e_i, \tag{12}$$

is a II-orthonormal local frame. Hence,

$$\begin{aligned}\mathrm{tr}_{\mathrm{II}}(\mathrm{Ric}) &= \sum_{i=1}^{2} \mathrm{Ric}(V_i, V_i) \\ &= -\sum_{i=1}^{2}\frac{1}{\lambda_i}\overline{\mathrm{Ric}}(e_i,e_i) - \sum_{i=1}^{2}\frac{1}{\lambda_i}\bar{g}(\bar{R}(e_i,N)e_i,N) + \mathrm{tr}\,(A).\end{aligned} \tag{13}$$

Further, we have that $\overline{\mathrm{Ric}}(e_i, e_i) = \bar{K} - \bar{g}(\bar{R}(e_i, N)e_i, N)$, $i = 1, 2$. Putting this in (13), and using (3), the proof ends. □

Now, we define the tangent vector field $\mathrm{S}(N)$ on M through the formula

$$\overline{\mathrm{Ric}}(N,X) = \mathrm{II}(\mathrm{S}(N),X),$$

for all $X \in \mathfrak{X}(M)$. If we take the trace of the Codazzi equation (2), with $\{e_1, e_2\}$ a local frame as above and $X \in \mathfrak{X}(M)$, we find

$$\begin{aligned}\overline{\mathrm{Ric}}(N,X) &= \sum_{i=1}^{2}\left\{\bar{g}((\nabla_X A)(e_i),e_i) - \bar{g}((\nabla_{e_i}A)(X),e_i)\right\} \\ &= \sum_{i=1}^{2}\left\{X\bar{g}(Ae_i,e_i) - 2\bar{g}(Ae_i,\nabla_X e_i) - \bar{g}(X,(\nabla_{e_i}A)(e_i))\right\} \\ &= X\mathrm{tr}\,(A) - \bar{g}(X,\mathrm{div}(A)) = \mathrm{II}(X,\nabla^{\mathrm{II}}\mathrm{tr}\,(A)) + \mathrm{II}(X,A^{-1}\mathrm{div}(A)),\end{aligned}$$

and therefore $\mathrm{S}(N) = \nabla^{\mathrm{II}}\mathrm{tr}\,(A) + A^{-1}\mathrm{div}(A)$. If we now use $A\nabla^{\mathrm{II}} f = -\nabla f$, for every smooth function f, we can write

$$A\mathrm{S}(N) = \mathrm{div}(A - \mathrm{tr}\,(A)I). \tag{14}$$

This equation is known as the momentum constraint in the initial-value problem of General Relativity (see [5] for a discussion of this equation in the 3-dimensional case). In the particular case of a totally umbilical spacelike surface with $A = \rho I$, there holds that

$$\mathrm{S}(N) = \nabla^{\mathrm{II}}\rho. \tag{15}$$

Lemma 3.2. *The trace of the tensor* $\widehat{Q}_1$ *with respect to* II *is given by*

$$\mathrm{tr}_{\mathrm{II}}(\widehat{Q}_1) = -\mathrm{div}_{\mathrm{II}}\left(\frac{A\mathrm{S}(N)}{\det(A)}\right).$$

Proof. Let $\{V_1, V_2\}$ be a II-orthonormal local frame. Then

$$\begin{aligned}
\mathrm{tr}_{\mathrm{II}}(\widehat{Q}_1) &= \sum_{i,j=1}^{2}\left\{\mathrm{II}\left(\left(\nabla^{\mathrm{II}}_{V_j}L\right)(V_i,V_i),V_j\right) - \mathrm{II}\left(\left(\nabla^{\mathrm{II}}_{V_i}L\right)(V_i,V_j),V_j\right)\right\}\\
&= \sum_{i,j=1}^{2} V_j\left\{\mathrm{II}(L(V_i,V_i),V_j) - \mathrm{II}(L(V_i,V_j),V_i)\right\}\\
&\quad + \sum_{i,j=1}^{2}\left\{\mathrm{II}(L(V_i,\nabla^{\mathrm{II}}_{V_j}V_j),V_i) - \mathrm{II}(L(V_i,V_i),\nabla^{\mathrm{II}}_{V_j}V_j)\right\}\\
&\quad + \sum_{i,j=1}^{2}\left\{\mathrm{II}(L(\nabla^{\mathrm{II}}_{V_i}V_j,V_i),V_j) - \mathrm{II}(L(\nabla^{\mathrm{II}}_{V_i}V_j,V_j),V_i)\right\}\\
&\quad + \sum_{i,j=1}^{2}\left\{\mathrm{II}(L(V_j,V_i),\nabla^{\mathrm{II}}_{V_i}V_j) - \mathrm{II}(L(V_j,\nabla^{\mathrm{II}}_{V_i}V_j),V_i)\right\}.
\end{aligned}$$

Applying (8) we have

$$\begin{aligned}
\mathrm{tr}_{\mathrm{II}}(\widehat{Q}_1) &= \sum_{i,j=1}^{2}\Big\{V_j\bar{g}(\bar{R}(V_i,N)V_i,V_j) + \bar{g}(\bar{R}(V_i,N)\nabla^{\mathrm{II}}_{V_j}V_j,V_i)\\
&\quad + \bar{g}(\bar{R}(\nabla^{\mathrm{II}}_{V_i}V_j,N)V_i,V_j) + \bar{g}(\bar{R}(V_j,N)V_i,\nabla^{\mathrm{II}}_{V_i}V_j)\Big\}.
\end{aligned}$$

If we now make use of the definition (4) of II we find

$$\begin{aligned}
\mathrm{tr}_{\mathrm{II}}(\widehat{Q}_1) &= -\sum_{j=1}^{2}\mathrm{II}\left(\nabla^{\mathrm{II}}_{V_j}\sum_{i=1}^{2}A^{-1}\bar{R}(V_i,N)V_i,V_j\right)\\
&\quad + \sum_{i,j=1}^{2}\left\{\bar{g}\left(\bar{R}(\nabla^{\mathrm{II}}_{V_i}V_j,N)V_i,V_j\right) + \bar{g}\left(\bar{R}(V_j,N)V_i,\nabla^{\mathrm{II}}_{V_i}V_j\right)\right\}.
\end{aligned} \tag{16}$$

Take a local frame $\{V_1, V_2\}$ as in (12). Then, for every vector field $X = X_1e_1 + X_2e_2$, we have

$$\begin{aligned}
\sum_{i=1}^{2}\mathrm{II}(A^{-1}\bar{R}(V_i,N)V_i,X) &= \frac{1}{\lambda_1}\bar{g}(\bar{R}(e_1,N)e_1,X) + \frac{1}{\lambda_2}\bar{g}(\bar{R}(e_2,N)e_2,X)\\
&= \frac{1}{\lambda_1}\overline{\mathrm{Ric}}(N,X_2e_2) + \frac{1}{\lambda_2}\overline{\mathrm{Ric}}(N,X_1e_1)\\
&= \frac{1}{\det(A)}\overline{\mathrm{Ric}}(N,AX) = \frac{1}{\det(A)}\mathrm{II}(A\mathrm{S}(N),X),
\end{aligned}$$

and therefore,

$$\sum_{i=1}^{2} A^{-1}\bar{R}(V_i, N)V_i = \frac{1}{\det(A)} A S(N). \tag{17}$$

Taking into account

$$\mathrm{II}(\nabla^{\mathrm{II}}_{V_1} V_1, V_2) = -\mathrm{II}(\nabla^{\mathrm{II}}_{V_1} V_2, V_1), \mathrm{II}(\nabla^{\mathrm{II}}_{V_2} V_1, V_2) = -\mathrm{II}(\nabla^{\mathrm{II}}_{V_2} V_2, V_1),$$
$$\mathrm{II}(\nabla^{\mathrm{II}}_{V_1} V_1, V_1) = \mathrm{II}(\nabla^{\mathrm{II}}_{V_2} V_1, V_1) = \mathrm{II}(\nabla^{\mathrm{II}}_{V_1} V_2, V_2) = \mathrm{II}(\nabla^{\mathrm{II}}_{V_2} V_2, V_2) = 0,$$

Eq. (16) becomes the one we are looking for. □

Next we consider, for each $X \in \mathfrak{X}(M)$, the operator $L_X := \nabla^{\mathrm{II}}_X - \nabla_X$. If we take a II-orthonormal local frame $\{V_1, V_2\}$, then the trace of L_X with respect to II is computed, using (8), as follows

$$\begin{aligned}\mathrm{tr}_{\mathrm{II}} L_X &= \sum_{i=1}^{2} \mathrm{II}(L(V_i, V_i), X) + \sum_{i=1}^{2} \bar{g}(\bar{R}(V_i, X)N, V_i) \\ &= \mathrm{II}(\mathrm{tr}_{\mathrm{II}} L, X) + \mathrm{II}(A^{-1} \sum_{i=1}^{2} \bar{R}(V_i, N)V_i, X).\end{aligned}$$

Finally, from (17) we find, for every $X \in \mathfrak{X}(M)$,

$$\mathrm{tr}_{\mathrm{II}} L_X = \mathrm{II}\left(\mathrm{tr}_{\mathrm{II}} L + \frac{AS(N)}{\det(A)}, X\right). \tag{18}$$

Lemma 3.3. *The trace of the tensor* $\widehat{Q}_2$ *with respect to* II *is given by*

$$\mathrm{tr}_{\mathrm{II}}(\widehat{Q}_2) = \|L\|^2_{\mathrm{II}} - \|\mathrm{tr}_{\mathrm{II}} L\|^2_{\mathrm{II}} - \mathrm{tr}_{\mathrm{II}} L_{\frac{AS(N)}{\det(A)}}, \tag{19}$$

whereby $\|L\|^2_{\mathrm{II}}$ *is the squared* II*-length of the difference tensor* L.

Proof. Let $\{V_1, V_2\}$ be a II-orthonormal local frame. Then, after using (8), we have

$$\begin{aligned}\mathrm{tr}_{\mathrm{II}}(\widehat{Q}_2) = \sum_{i,j=1}^{2} \big\{&\mathrm{II}(L(V_i, V_j), L(V_i, V_j)) - \bar{g}(\bar{R}(V_i, N)V_j, L(V_i, V_j)) \\ &- \mathrm{II}(L(V_i, V_i), L(V_j, V_j)) + \bar{g}(\bar{R}(V_j, N)V_j, L(V_i, V_i))\big\}.\end{aligned} \tag{20}$$

Note that the first term is $\|L\|^2_{\mathrm{II}}$ and the third $\|\mathrm{tr}_{\mathrm{II}} L\|^2_{\mathrm{II}}$. Further, the second term in (20) can be written as

$$\begin{aligned}\sum_{i,j=1}^{2} \bar{g}(\bar{R}(V_i, N)V_j, L(V_i, V_j)) = &-\frac{1}{2} \sum_{i,j=1}^{2} \mathrm{II}(A^{-1}\bar{R}(V_i, N)V_j, A^{-1}(\nabla_{V_i} A)(V_j)) \\ &+ \frac{1}{2} \sum_{i,j=1}^{2} \mathrm{II}(A^{-1}\bar{R}(V_i, N)V_j, A^{-1}\bar{R}(V_i, N)V_j).\end{aligned}$$

The first term is zero because of the symmetry properties $\bar{g}(\bar{R}(X, N)Y, Z) = -\bar{g}(\bar{R}(X, N)Z, Y)$ and $\bar{g}((\nabla_X A)(Y), Z) = \bar{g}((\nabla_X A)(Z), Y)$, while the last term becomes, using again the expression (12) for the local frame $\{V_1, V_2\}$,

$$\begin{aligned}\frac{1}{2} \sum_{i,j=1}^{2} \mathrm{II}(A^{-1}\bar{R}(V_i, N)V_j, A^{-1}\bar{R}(V_i, N)V_j) &= \frac{1}{2} \sum_{i,j,k=1}^{2} \left[\bar{g}(\bar{R}(V_i, N)V_j, V_k)\right]^2 \\ &= -\frac{1}{\lambda_1^2 \lambda_2} \left[\bar{g}(\bar{R}(e_1, N)e_1, e_2)\right]^2 - \frac{1}{\lambda_1 \lambda_2^2} \left[\bar{g}(\bar{R}(e_2, N)e_2, e_1)\right]^2\end{aligned}$$

$$= \frac{1}{(\det A)^2}\left\{\mathrm{II}(e_1,e_1)\left[\overline{\mathrm{Ric}}(N,e_1)\right]^2 + \mathrm{II}(e_2,e_2)\left[\overline{\mathrm{Ric}}(N,e_2)\right]^2\right\}$$
$$= \frac{1}{(\det A)^2}\|A\mathrm{S}(N)\|^2_{\mathrm{II}}.$$

Finally, the last term in (20) becomes, using (17),

$$\sum_{i,j=1}^{2}\bar{g}(\bar{R}(V_j,N)V_j, L(V_i,V_i)) = -\mathrm{II}\left(\sum_{j=1}^{2}A^{-1}\bar{R}(V_j,N)V_j, \mathrm{tr}_{\mathrm{II}}L\right)$$
$$= -\mathrm{II}\left(\frac{A\mathrm{S}(N)}{\det(A)}, \mathrm{tr}_{\mathrm{II}}L\right).$$

Collecting all the terms and using (18), we find (19). □

From formula (11), and bearing in mind the three lemmas above we get

Theorem 3.4. *Let M be a spacelike surface with Gauss–Kronecker curvature* $\det(A) > 0$ *in a* 3D *spacetime* $\bar{M}$. *Then, the Gauss curvature of the metric* II *satisfies*

$$2K^{\mathrm{II}} = -\frac{\mathrm{tr}\,(A)}{\det(A)}K - \mathrm{div}_{\mathrm{II}}\left(\frac{A\mathrm{S}(N)}{\det(A)}\right) + \|L\|^2_{\mathrm{II}} - \|\mathrm{tr}_{\mathrm{II}}L\|^2_{\mathrm{II}} - \mathrm{tr}_{\mathrm{II}}L_{\frac{A\mathrm{S}(N)}{\det(A)}}, \tag{21}$$

where $\|L\|^2_{\mathrm{II}}$ *is the squared* II*-length of* L *and* $\mathrm{tr}_{\mathrm{II}}L_{\frac{A\mathrm{S}(N)}{\det(A)}}$ *is given by* (18).

Note that this formula widely generalizes formula (7) in [3].

Remark 3.5. Choose a II-orthonormal local frame as in (12). We can then express $\mathrm{tr}_{\mathrm{II}}L$ in terms of $\det(A)$ and $\mathrm{S}(N)$ as follows. From the Codazzi equation we have that

$$\frac{1}{\det(A)}\mathrm{II}(X, \nabla^{\mathrm{II}}\det(A)) = \sum_{i=1}^{2}\{\mathrm{II}(A^{-1}(\nabla_{V_i}A)(V_i), X) + \mathrm{II}(A^{-1}\bar{R}(V_i,N)V_i, X)\},$$

for all $X \in \mathfrak{X}(M)$, and hence, using (7) and (17), we have

$$\mathrm{tr}_{\mathrm{II}}(L) = \frac{1}{2\det(A)}\nabla^{\mathrm{II}}\det(A) - \frac{A\mathrm{S}(N)}{\det(A)}. \quad \square \tag{22}$$

Remark 3.6. With the help of the local frame (12) we can compute the last three terms in (21) explicitly, and thus we obtain the following expression for K^{II} in terms of the principal curvatures $\{\lambda_1, \lambda_2\}$ and principal directions $\{e_1, e_2\}$ of the spacelike surface,

$$\begin{aligned}2K^{\mathrm{II}} &= -\frac{\mathrm{tr}\,(A)}{\det(A)}K - \mathrm{div}_{\mathrm{II}}\left(\frac{A\mathrm{S}(N)}{\det(A)}\right)\\ &\quad -\frac{1}{2\det(A)}\left\{e_1\left(\log\left(\frac{\lambda_2}{\lambda_1}\right)\right)\left[e_1(\lambda_2) - \overline{\mathrm{Ric}}(N,e_1)\right]\right.\\ &\quad \left. + e_2\left(\log\left(\frac{\lambda_1}{\lambda_2}\right)\right)\left[e_2(\lambda_1) - \overline{\mathrm{Ric}}(N,e_2)\right]\right\}.\end{aligned} \tag{23}$$

Note that this formula is an extension of the formula for K^{II} obtained in [8, p. 18] in the case of a surface in $\mathbb{R}^3$ and of formula (12) in [10] in the case of surfaces in ambient spaces of constant sectional curvature. □

4. Main results

In this section we give several characterizations of compact totally umbilical spacelike surfaces, with signed Gauss curvature, based on assumptions on the second fundamental form II.

Theorem 4.1. *Let M be a compact spacelike surface in a* 3D *spacetime $\bar{M}$, with non-zero Euler–Poincaré characteristic $\chi(M)$,* $\det(A) > 0$ *and K signed (i.e. $K > 0$ or < 0). Then, M is totally umbilical if and only if*

$$\chi(M)\int_M \{\|L\|^2_{\mathrm{II}} - \|\mathrm{tr}_{\mathrm{II}} L\|^2_{\mathrm{II}} - \mathrm{tr}_{\mathrm{II}} L_{\frac{A\mathrm{S}(N)}{\det(A)}}\}\,\mathrm{d}\Omega_{\mathrm{II}} \geq 0.$$

Proof. Assume first that $K > 0$. From the Gauss–Bonnet theorem it follows that $\chi(M) > 0$. Then, integrating (21) we obtain

$$\int_M K^{\mathrm{II}}\,\mathrm{d}\Omega_{\mathrm{II}} \geq -\int_M \frac{\mathrm{tr}\,(A)}{2\det(A)} K\,\mathrm{d}\Omega_{\mathrm{II}}.$$

The Euler inequality states that $-\mathrm{tr}\,(A) \geq 2\sqrt{\det(A)}$, with equality holding at every point if and only if the surface is totally umbilical. Hence, since $K > 0$, we have

$$\int_M K^{\mathrm{II}}\,\mathrm{d}\Omega_{\mathrm{II}} \geq \int_M \frac{K}{\sqrt{\det(A)}}\,\mathrm{d}\Omega_{\mathrm{II}}.$$

From the Gauss–Bonnet theorem, and using the relation $\mathrm{d}\Omega_{\mathrm{II}} = \sqrt{\det(A)}\mathrm{d}\Omega$ between the area elements of M with respect to II and $\bar{g}$ respectively, we have

$$2\pi\chi(M) = \int_M K^{\mathrm{II}}\,\mathrm{d}\Omega_{\mathrm{II}} \geq \int_M \frac{K}{\sqrt{\det(A)}}\,\mathrm{d}\Omega_{\mathrm{II}} = \int_M K\,\mathrm{d}\Omega = 2\pi\chi(M),$$

and thus, equality holds in the Euler inequality. The case $K < 0$ follows analogously. Conversely, if M is totally umbilical, then it follows from (23) that the integrand in the statement is identically zero. □

Remark 4.2. Note that the assumption on the sign of K is needed to be able to apply the Euler inequality in every point of M. This condition, however, is not the most general. It is sufficient, for example, to assume that $K \geq 0$ or $K \leq 0$ and that the set $\{p \in M \mid K(p) = 0\}$ has no interior point. In fact, if it is assumed that $K \geq 0$, the equality $\mathrm{tr}\,(A) = -2\sqrt{\det(A)}$ is satisfied on the open set $M_+ = \{p \in M \mid K(p) > 0\}$ and thus, by continuity, also on the closure of M_+ in M. As the set $\{p \in M \mid K(p) = 0\}$ has no interior points, M_+ is dense in M and the equality $\mathrm{tr}\,(A) = -2\sqrt{\det(A)}$ holds on all of M. The case $K \leq 0$ is analogous. □

In the very particular case that the 3D spacetime is the de Sitter space $\mathbf{S}^3_1$ we have $\bar{K} = 1$, and $\det(A) > 0$ is here equivalent to $K < 1$. As $\mathbf{S}^3_1$ is a space of constant curvature, there further holds that $\mathrm{S}(N) = 0$. Hence, if K is constant, $\det(A)$ is constant and thus $\mathrm{tr}_{\mathrm{II}} L = 0$. Since every compact spacelike surface in the 3-dimensional de Sitter space is topologically a sphere, it follows from the Gauss–Bonnet theorem that, if K is constant, then $K > 0$. Thus Theorem 4.1 generalizes the following result in [3] (and consequently [12, Proposition 4.2]).

Corollary 4.3. *Every compact spacelike surface of the de Sitter space $\mathbf{S}^3_1$, with constant Gauss curvature $K < 1$, is a totally umbilical round sphere.*

Now, let us look into the assumption on L in Theorem 4.1.

Lemma 4.4. *Let M be a spacelike surface with Gauss–Kronecker* $\det(A) > 0$ *in a* 3D *spacetime $\bar{M}$. Then for the difference tensor L, defined in (6), we always have the inequality $\|L\|^2_{\mathrm{II}} \geq \frac{1}{2}\|\mathrm{tr}_{\mathrm{II}} L\|^2_{\mathrm{II}}$, and equality holds if and only if $L(X,Y) = -\mathrm{II}(X,Y)\frac{A\mathrm{S}(N)}{\det(A)}$.*

Proof. Let $\{V_1, V_2\}$ be a II-orthonormal local frame and denote by L^k_{ij} the function $\bar{g}(L(V_i,V_j),V_k)$. For every k fixed, (L^k_{ij}) is a symmetric 2×2 matrix and thus the Schwartz inequality gives $\left(\sum_{i=1}^2 L^k_{ii}\right)^2 \leq 2\sum_{i,j=1}^2 (L^k_{ij})^2$. If we sum over k we obtain the announced inequality, and equality holds if and only if there exist smooth functions λ^k such that $L^k_{ij} = \lambda^k \delta_{ij}$, for all $i, j, k \in \{1,2\}$. Thus, equality holds if and only if there exists a vector field $T = \sum_{k=1}^2 \lambda^k V_k$ satisfying $L(X,Y) = \mathrm{II}(X,Y)T$, for all $X, Y \in \mathfrak{X}(M)$. Finally, we find that T must be equal to $-\frac{A\mathrm{S}(N)}{\det(A)}$. □

We then have another consequence of Theorem 4.1.

Corollary 4.5. *Let M be a compact spacelike surface in a* 3D *spacetime* $\bar{M}$. *If* $\det(A) > 0$, $\|L\|^2_{\mathrm{II}} = \frac{1}{2}\|\mathrm{tr}_{\mathrm{II}}L\|^2_{\mathrm{II}}$ *and* $K < 0$, *then M is totally umbilical with* $A = \rho I$, ρ *constant.*

Proof. If $\|L\|^2_{\mathrm{II}} = \frac{1}{2}\|\mathrm{tr}_{\mathrm{II}}L\|^2_{\mathrm{II}}$ it follows from Lemma 4.4 that $\mathrm{tr}_{\mathrm{II}}L = -2\frac{A\mathrm{S}(N)}{\det(A)}$ and from (18) that $\mathrm{tr}_{\mathrm{II}}L_{\frac{A\mathrm{S}(N)}{\det(A)}} = -\|\frac{A\mathrm{S}(N)}{\det(A)}\|^2_{\mathrm{II}}$. Therefore, we have

$$\|L\|^2_{\mathrm{II}} - \|\mathrm{tr}_{\mathrm{II}}L\|^2_{\mathrm{II}} - \mathrm{tr}_{\mathrm{II}}L_{\frac{A\mathrm{S}(N)}{\det(A)}} = -\left\|\frac{A\mathrm{S}(N)}{\det(A)}\right\|^2_{\mathrm{II}} \leq 0.$$

Since $K < 0$ it follows from the Gauss–Bonnet theorem that $\chi(M) < 0$, and then, from Theorem 4.1, we have that M must be totally umbilical. Now, since every totally umbilical spacelike surface in a 3D spacetime has $\mathrm{tr}_{\mathrm{II}}L = 0$ it follows that $\mathrm{S}(N) = 0$. Therefore, from (15), we find that ρ must be constant. □

Finally, we can give in the following result a converse of Lemma 2.4.

Theorem 4.6. *Let M be a compact spacelike surface in a* 3D *spacetime* $\bar{M}$, *with* $\bar{K} > K > 0$. *Then, M is totally umbilical if and only if* $\mathrm{tr}_{\mathrm{II}}L = 0$.

Proof. Choose a $\bar{g}$-orthonormal local frame $\{e_1, e_2\}$ which diagonalizes A, i.e., $Ae_i = \lambda_i e_i$, $i = 1, 2$. From (14), an easy computation gives

$$A\mathrm{S}(N) = \{-e_1(\lambda_2) + (\lambda_2 - \lambda_1)\bar{g}(e_1, \nabla_{e_2}e_2)\}e_1 + \{-e_2(\lambda_1) + (\lambda_1 - \lambda_2)\bar{g}(e_2, \nabla_{e_1}e_1)\}e_2,$$

and thus we have that

$$e_1(\lambda_2) - \overline{\mathrm{Ric}}(N, e_1) = (\lambda_2 - \lambda_1)\bar{g}(e_1, \nabla_{e_2}e_2),$$
$$e_2(\lambda_1) - \overline{\mathrm{Ric}}(N, e_2) = (\lambda_1 - \lambda_2)\bar{g}(e_2, \nabla_{e_1}e_1).$$

From (22) we have that the condition $\mathrm{tr}_{\mathrm{II}}L = 0$ is equivalent to $A\mathrm{S}(N) = \frac{1}{2}\nabla^{\mathrm{II}}\det(A)$, which gives the two equations

$$(\lambda_2 - \lambda_1)\bar{g}(e_1, \nabla_{e_2}e_2) = \frac{1}{2}\lambda_2 e_1\left(\log\left(\frac{\lambda_2}{\lambda_1}\right)\right),$$
$$(\lambda_1 - \lambda_2)\bar{g}(e_2, \nabla_{e_1}e_1) = \frac{1}{2}\lambda_1 e_2\left(\log\left(\frac{\lambda_1}{\lambda_2}\right)\right).$$

Now, from (23) we have that $\|L\|^2_{\mathrm{II}} - \|\mathrm{tr}_{\mathrm{II}}L\|^2_{\mathrm{II}} - \mathrm{tr}_{\mathrm{II}}L_{\frac{A\mathrm{S}(N)}{\det(A)}}$ must be equal to

$$\frac{1}{4\det(A)}\left\{-\lambda_2\left[e_1\left(\log\left(\frac{\lambda_2}{\lambda_1}\right)\right)\right]^2 - \lambda_1\left[e_2\left(\log\left(\frac{\lambda_1}{\lambda_2}\right)\right)\right]^2\right\},$$

which is non-negative. The proof is then completed using Theorem 4.1. □

Remark 4.7. From the Einstein equation $\overline{\mathrm{Ric}} - \frac{1}{2}\bar{S}\bar{g} = \kappa\overline{T}$, with κ a positive constant and $\overline{T}$ the energy–momentum tensor, it follows that $\bar{K} = \overline{\mathrm{Ric}}(N, N) + \frac{1}{2}\bar{S} = \kappa\overline{T}(N, N)$. Thus $\bar{K}$ is the relative local energy of an observer whose rest space is M which is moving with 3-velocity N. Then, the condition $\bar{K} > K > 0$ can be interpreted as giving a lower bound to the total energy, $\int_M \bar{K}\,\mathrm{d}\Omega > \int_M K\,\mathrm{d}\Omega = 4\pi$. □

Acknowledgements

J.A. Aledo was partially supported by MCYT-FEDER Grant No MTM2004-02746. A. Romero was partially supported by MEC-FEDER Grant MTM2004-04934-C04-01.

References

[1] J.A. Aledo, L.J. Alías, A. Romero, Integral formulas for compact space-like hypersurfaces in de Sitter space: Applications to the case of constant higher order mean curvature, J. Geom. Phys. 31 (1999) 195–208.
[2] J.A. Aledo, L.J. Alías, A. Romero, A new proof of Liebmann classical theorem for surfaces in space forms, Rocky Mountain J. Math. 35 (2005) 1811–1824.
[3] J.A. Aledo, A. Romero, Compact spacelike surfaces in the 3-dimensional de Sitter space with non-degenerate second fundamental form, Differential Geom. Appl. 19 (2003) 97–111.
[4] L.J. Alías, A. Romero, M. Sánchez, Uniqueness of complete spacelike hypersurfaces of constant mean curvature in Generalized Robertson–Walker spacetimes, Gen. Relativity Gravitation 27 (1995) 71–84.
[5] L. Andersson, V. Moncrief, A.J. Tromba, On the global evolution problem in $2 + 1$ gravity, J. Geom. Phys. 23 (1997) 191–205.
[6] C. Carlip, Conformal field theory, $(2 + 1)$-dimensional gravity and the BTZ black hole, arXiv:gr-qc/0503022 v3.
[7] C. Carlip, Quantum Gravity in $2 + 1$ dimensions: The case of the Closed Universe, Living Rev. Relativity 8 (2005) 1–63.
[8] É Cartan, Les surfaces qui admettent une seconde forme fondamentale donnée, Bull. Sci. Math. 67 (1943) 8–32.
[9] N. Hicks, Linear perturbations of connections, Michigan Math. J. 12 (1965) 389–397.
[10] T. Klotz-Milnor, The curvature of $\alpha \mathrm{I} + \beta \mathrm{II} + \gamma \mathrm{III}$ on a surface in a 3-manifold of constant curvature, Michigan Math. J. 22 (1975) 247–255.
[11] T. Koufogiorgos, T. Hasanis, A characteristic property of the sphere, Proc. Amer. Math. Soc. 67 (1977) 303–305.
[12] H. Li, Global rigidity theorems of hypersurfaces, Ark. Mat. 35 (1997) 327–351.
[13] V. Moncrief, Reduction of the Einstein equation in $2 + 1$ dimensions to a Hamiltonian system over Teichmüller space, J. Math. Phys. 30 (1989) 2907–2914.
[14] R. Schneider, Closed convex hypersurfaces with second fundamental form of constant curvature, Proc. Amer. Math. Soc. 35 (1972) 230–233.
[15] U. Simon, Characterizations of the sphere by the curvature of the second fundamental form, Proc. Amer. Math. Soc. 55 (1976) 382–384.